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HW 4.6 Advanced Trigonometric Identities: Sum and Product Identities:

$\sin(a+b)=\sin a\cos b+\sin b\cos a$, $a+b=A$

$\sin(a-b)=\sin a\cos b-\sin b\cos a$, $a-b=B$

So $a=\frac{1}{2}(A+B)$, $b=\frac{1}{2}(A-B)$

PRODUCT TO SUM IDENTIES

$$\sin a\cos b=\tfrac{1}{2}\left[\sin(a+b)+\sin(a-b)\right]$$
$$\cos a\sin b=\tfrac{1}{2}\left[\sin(a+b)-\sin(a-b)\right]$$
$$\cos a\cos b=\tfrac{1}{2}\left[\cos(a+b)+\cos(a-b)\right]$$
$$\sin a\sin b=-\tfrac{1}{2}\left[\cos(a+b)-\cos(a-b)\right]$$

SUM TO PRODUCT IDENTITIES

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\sin A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$
$$\cos A+\cos B=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\cos A-\cos B=-2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

1. Evaluate each of the following using the formulas above: (No Calculators)

a) $2\sin 45^\circ\cos 15^\circ$

$a=45^\circ$, $b=15^\circ$ $2\sin a\cos b=\sin(a+b)+\sin(a-b)$

$=\sin(45+15)+\sin(45-15)$

$=\frac{\sqrt3}{2}+\frac{1}{2}=\boxed{\frac{\sqrt3+1}{2}}$

b) $2\sin 82.5^\circ\cos 37.5^\circ$

$=\sin(82.5+37.5)+\sin(82.5-37.5)$

$=\sin 120^\circ+\sin 45^\circ=\frac{\sqrt3}{2}+\frac{\sqrt2}{2}=\boxed{\frac{\sqrt3+\sqrt2}{2}}$

c) $2\sin 127.5^\circ\sin 97.5^\circ$

$\cos(a+b)=\cos a\cos b-\sin a\sin b$

$-\cos(a-b)=\cancel{\cos a\cos b}+\sin a\sin b$

$\cos(a+b)-\cos(a-b)=-2\sin a\sin b$

$=-\cos(225^\circ)+\cos 30^\circ=\frac{\sqrt2}{2}+\frac{\sqrt3}{2}=\boxed{\frac{\sqrt3+\sqrt2}{2}}$

d) $\cos 15^\circ$

Approach 1

$\cos\left(\frac{\theta}{2}\right)=\pm\sqrt{\frac{\cos\theta+1}{2}}$

$\cos\left(\frac{30}{2}\right)=\pm\sqrt{\frac{\cos 30+1}{2}}=\pm\sqrt{\frac{\frac{\sqrt3}{2}+1}{2}}=\boxed{\frac{\sqrt{\sqrt3+2}}{2}}$

Approach 2

$\cos(15^\circ)=\cos(45-30)=\cos45\cos30-\sin45\sin30$

$=\frac{\sqrt2}{2}\cdot\frac{\sqrt3}{2}-\frac{\sqrt2}{2}\cdot\frac{1}{2}=\boxed{\frac{\sqrt6-\sqrt2}{4}}$

e) $\sin 105^\circ+\sin 15^\circ$

$\sin 105^\circ+\sin 15^\circ=2\sin 60^\circ\cos 45^\circ$

$=\cancel{2}\cdot\frac{\sqrt3}{\cancel{2}}\cdot\frac{\sqrt2}{2}=\boxed{\frac{\sqrt6}{2}}$

f) $\cos 465^\circ+\cos 165^\circ$

$=2\sin 315^\circ\cos 150^\circ$

$=\cancel{2}\cdot\frac{\sqrt2}{\cancel{2}}\cdot\frac{-\sqrt3}{2}=\boxed{\frac{-\sqrt6}{2}}$

g) $\sin 75^\circ-\sin 15^\circ$

$=2\cos 45\cdot\sin 30$

$=\cancel{2}\cdot\frac{\sqrt2}{\cancel{2}}\cdot\frac{1}{2}=\boxed{\frac{\sqrt2}{2}}$

h) $\frac{\cos 75^\circ+\cos 15^\circ}{\sin 75^\circ-\sin 15^\circ}$

$=\frac{\cancel{2\cos45}\cdot\cos 30}{\cancel{2\cos45}\cdot\sin 30}=\frac{\frac{\sqrt3}{2}}{\frac{1}{2}}=\boxed{\sqrt3}$

2. Match each of the following sums from the left with the corresponding product on the right:

a) $\sin 4x + \sin 2x$ $\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$	vi	i) $2\cos 4x \cos 2x$
b) $\sin 7x - \sin 3x$ $\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A+B)$	iv	ii) $2\sin 35° \cos 15°$
c) $\cos 6x + \cos 2x$ $\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$	i	iii) $-2\sin 75° \sin 5°$
d) $\cos\frac{3x}{2} - \cos\frac{9x}{2}$ $\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$	viii	iv) $2\cos 5x \sin 2x$
e) $\cos 65° + \cos 15°$ $\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$	vii	v) $2\cos 55° \sin 20°$
f) $\sin 50° + \sin 20°$ $\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$	ii	vi) $2\sin 3x \cos x$
g) $\cos 80° - \cos 70°$ $\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$	iii	vii) $2\cos 40° \cos 25°$
h) $\sin 75° - \sin 35°$ $\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$	v	viii) $2\sin 3x \sin\frac{3x}{2}$

3. Prove the following Identities:

a) $\dfrac{\sin a + \sin 3a}{\cos a + \cos 3a} = \tan 2a$ $= \dfrac{\cancel{2}\sin 2a \cancel{\cos a}}{\cancel{2}\cos 2a \cancel{\cos a}} = \tan 2a = RHS$ // Sum to product identities used.	b) $\dfrac{\sin 2a + \sin 4a}{\cos 2a + \cos 4a} = \tan 3a$ $= \dfrac{2\sin 3a \cancel{\cos a}}{2\cos 3a \cancel{\cos a}} = \tan 3a = RHS$ // sum to product identities used.

c) $\dfrac{\sin a+\sin b}{\sin a-\sin b}=\dfrac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}(a-b)}$

$$=\frac{\cancel{2}\sin\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{\cancel{2}\cos\frac{1}{2}(a+b)\sin\frac{1}{2}(a-b)}$$

sum to product identity

$\tan\theta=\frac{\sin\theta}{\cos\theta}$

$$=\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}(a-b)}=\text{RHS}$$

NPV: $\sin a\neq\sin b$

$\tan\frac{1}{2}(a-b)\neq 0$

d) $\dfrac{\cos a+\cos b}{\cos a-\cos b}=-\cot\frac{1}{2}(a-b)\cot\frac{1}{2}(a+b)$

$$=\frac{\cancel{2}\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{-\cancel{2}\sin\frac{1}{2}(a+b)\sin\frac{1}{2}(a-b)}$$

$$=-\cot\tfrac{1}{2}(a+b)\cot(a-b)=\text{RHS}$$

NPV: $\cos a\neq\cos b$

$\sin\frac{1}{2}(a+b)\neq 0$

e) $\cos a+\cos 2a+\cos 3a=\cos 2a(1+\cos a)$

$=\cos a+\cos 3a+\cos 2a$

$=2\cos 2a\cos a+\cos 2a$

$=\cos 2a(2\cos a+1)$

Not an identity

f) $\dfrac{\sin 3a+\sin 5a+\sin 7a+\sin 9a}{\cos 3a+\cos 5a+\cos 7a+\cos 9a}=\tan 6a$

$$\frac{(\sin 3a+\sin 5a)+(\sin 7a+\sin 9a)}{(\cos 3a+\cos 5a)+(\cos 7a+\cos 9a)}=\tan 6a$$

$$\frac{\cancel{2}\sin 4a\cancel{\cos a}+\cancel{2}\sin 8a\cancel{\cos a}}{\cancel{2}\cos 4a\cancel{\cos a}+\cancel{2}\cos 8a\cancel{\cos a}}$$

~~double angle identity~~

$$=\frac{\sin 4a+\sin 8a}{\cos 4a+\cos 8a}$$

double angle identity

$$=\frac{\cancel{2}\sin 6a\cancel{\cos 2a}}{\cancel{2}\cos 6a\cancel{\cos 2a}}=\tan 6a=\text{RHS}$$

4. Prove or Evaluate each of the following:

$(\cos 20°)(\cos 40°)(\cos 80°) = \frac{1}{8}$ $\cos(a-b) = \cos a\cos b + \sin a \sin b$ $\cos(a+b) = \cos a\cos b - \sin a\sin b$ $\cos(a-b) + \cos(a+b) = 2\cos a\cos b$ $= \frac{1}{2}[\cos 20 + \cos 60]\cdot\cos 80$ $= \frac{1}{2}(\cos 20 + \frac{1}{2})\cdot \cos 80$ $= \frac{1}{2}\left[\cos 20\cos 80 + \frac{\cos 80}{2}\right] = \frac{1}{2}\left[\frac{1}{2}(\cos 60 + \cos 100) + \frac{\cos 80}{2}\right]$ $= \frac{1}{4}\left[\frac{1}{2} + \cancel{\cos 80} - \cancel{\cos 80}\right] = \frac{1}{8} = \text{RHS}$	$\frac{(\cos 4x - \cos 2x)}{2\sin 3x} = -\sin x \quad [\sin 3x \neq 0]$ $= \frac{-\cancel{2}\cancel{\sin 3x}\sin x}{\cancel{2\sin 3x}} = -\sin x = \text{RHS}$ Sum to product identity
$\underbrace{\cos 130° + \cos 110°} + \cos 10° = 0$ $= 2\cos 120\cos 10 + \cos 10$ Sum to product identity $= \cos 10(2\cos 120 + 1)$ $= \cos 10\left(2\cdot -\frac{1}{2} + 1\right)$ $= 0 = \text{RHS}$	$\cos 220° + \cos 100° + \cos 20° = 0$ $= 2\cos 160\cos 60 + \cos 20$ Sum to product identity $= -2\cos 20\cos 60 + \cos 20$ $\cos(\theta) = -\cos(180-\theta)$ $= \cos 20(-2\cos 60 + 1)$ $= \cos 20\left(-2\cdot\frac{1}{2} + 1\right)$ $= 0$

Prove the following Identity:

$$1 + \cos 2\theta + \cos 4\theta + \cos 6\theta = 4\cos\theta\cos 2\theta\cos 3\theta$$

$= 1 + 2\cos 3\theta\cos\theta + \cos 6\theta$ sum to product identity

$= \cancel{1} + 2\cos 3\theta\cos\theta + 2\cos^2 3\theta - \cancel{1}$ double angle identity

$= 2\cos 3\theta(\cos\theta + \cos 3\theta) = 2\cos 3\theta(2\cos 2\theta\cos\theta) = 4\cos\theta\cos 2\theta\cos 3\theta = \text{RHS}$

5. Challenge: Evaluate the following using trig. identities:

$$\frac{(\sin 13° + \sin 47°) + (\sin 73° + \sin 107°)}{\cos 17°}$$

$$\frac{2\sin 30\cancel{\cos 17} + 2\sin 90\cancel{\cos 17}}{\cancel{\cos 17}} = 2(\sin 30 + \sin 90) = 2\left(\frac{1}{2} + 1\right) = \boxed{3}$$

Sum to product identity!